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LARGE DYNAMIC DEFORMATIONS CAUSED BY A FORCE TRAVELING ON AN EXTENSIBLE STRING

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Abstract—An infinitely long, perfectly flexible string is subjected to two concentrated forces which travel along the string with constant speed in opposite directions from a common starting point. The ensuing motions and deformations of the string are described. The description takes into account large deformations, changes in string tension, and the transverse and longitudinal waves which propagate.

1. INTRODUCTION

A TECHNIQUE commonly used to study dynamic loading of structural elements is to give one surface of the element an initial, uniformly distributed, impulsive velocity and find the subsequent response. This type of loading is often produced in the laboratory (Humphreys [1]; Florence and Firth [2], for example) by ignition of a layer of sheet explosive placed on the element. Because the burning of the explosive progresses from the ignition point at a finite rate, the question arises as to what extent this procedure simulates a uniform distribution of impulsive velocity. Some answers may be obtained by consideration of the response of a very simple structure—an infinitely long perfectly flexible string—to two traveling concentrated forces which represent the two detonation fronts progressing in opposite directions from the point of ignition. This may then be compared with the response of the string to a truly uniform initial velocity distribution.

Kanninen and Florence [3] solved the problem of an infinitely long perfect string subjected to two concentrated transverse forces which travel along the string at constant speed in opposite directions from a common starting point. They considered the deformation of the string to be small with the string tension a constant. A solution to the same problem is given here which allows for large deformations, changes in string tension, and the propagation of longitudinal waves in the string.

It will be shown that the solution to the problem may be obtained by combining the kinematic and dynamic requirements on the transverse wave which occurs under the action of a traveling force with the solution to the field equations governing wave propagation in the string under no external forces. The solution to this latter problem was presented in connection with a study of nonlinear wave propagation in strings [4], but will be presented here using an approach somewhat different from that used in [4].

2. RESPONSE OF THE STRING UNDER NO EXTERNAL FORCES

Let X be the position of a particle of the string in the undeformed state, t the time, x and y horizontal and vertical particle displacements, σ the nominal (engineering) stress, ρ the mass density in the undeformed state, λ the ratio of the current length of a segment to its undeformed length, and ψ the angle the tangent to a segment makes with the initial longitudinal axis. The equations of motion are

$$\rho\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial X}(\sigma \sin \psi) \tag{1}$$

$$\rho\frac{\partial^2 x}{\partial t^2} = \frac{\partial}{\partial X}(\sigma \cos \psi). \tag{2}$$

From the kinematics of deformation,

$$\frac{\partial x}{\partial X} = \lambda \cos \psi - 1 \tag{3}$$

$$\frac{\partial y}{\partial X} = \lambda \sin \psi. \tag{4}$$

Transformation of these expressions into ones involving u and v, particle velocities respectively along and transverse to a wire segment, leads to

$$\frac{\partial u}{\partial t} - v\frac{\partial \psi}{\partial t} - \frac{1}{\rho}\frac{\mathrm{d}\sigma}{\mathrm{d}\lambda}\frac{\partial \lambda}{\partial X} = 0 \tag{5}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial \psi}{\partial t} - \frac{\sigma}{\rho}\frac{\partial \psi}{\partial X} = 0 \tag{6}$$

$$\frac{\partial u}{\partial X} - v\frac{\partial \psi}{\partial X} - \frac{\partial \lambda}{\partial t} = 0 \tag{7}$$

$$\frac{\partial v}{\partial X} + u\frac{\partial \psi}{\partial X} - \lambda\frac{\partial \psi}{\partial t} = 0. \tag{8}$$

In writing equation (5), it has been assumed that the behavior of the material of the string is governed by a constitutive equation of form

$$\sigma = \sigma(\lambda) \tag{9a}$$

in which

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\lambda} > 0 \qquad \frac{\mathrm{d}^2\sigma}{\mathrm{d}\lambda^2} < 0 \quad \text{for } \lambda \geq 1. \tag{9b}$$

The second requirement in equation (9b) may be removed if the longitudinal shock waves which would result when it is violated are accounted for. This is commonly done whenever (9a) incorporates a linear elastic region. Both nonlinear elastic and plastic deformation may be taken into account in (9a).

A similarity solution of these equations may be constructed. Assume that each of the four dependent variables appearing in equations (5)–(8) are functions only of

$$\eta = \frac{X}{t}$$

so that the two partial differentiation operations may be expressed as

$$\frac{\partial}{\partial X} = \frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}\eta} \qquad \frac{\partial}{\partial t} = -\frac{\eta}{t}\frac{\mathrm{d}}{\mathrm{d}\eta}.$$

Symbolizing differentiation with respect to η by a prime, the equations become

$$u' - v\psi' + \eta\lambda' = 0 \tag{10}$$

$$\eta u' - \eta v\psi' + c^2\lambda' = 0 \tag{11}$$

$$v' + (u + \eta\lambda)\psi' = 0 \tag{12}$$

$$\eta v' + (\eta u + \lambda\bar{c}^2)\psi' = 0 \tag{13}$$

where

$$c^2 = \frac{1}{\rho}\frac{\mathrm{d}\sigma}{\mathrm{d}\lambda} \qquad \bar{c}^2 = \frac{\sigma}{\rho\lambda} \tag{14}$$

Substituting equations (10) and (12), obtain in place of equations (11) and (13)

$$(\eta^2 - c^2)\lambda' = 0 \tag{15}$$

$$(\eta^2 - \bar{c}^2)\psi' = 0. \tag{16}$$

Equations (10), (12), (15) and (16) may be interpreted as follows. The exceptional case, $c = \bar{c}$, will be omitted.

(a) If $\eta \neq \pm\bar{c}$, ψ and v are constant.
(b) If $\eta \neq \pm c$, λ is constant, and unless $\eta = \pm\bar{c}$, u is also constant.
(c) When $\eta = \pm c$, λ and u may be discontinuous and furthermore,

$$u = \mp\int c\,\mathrm{d}\lambda \tag{17}$$

(d) When $\eta = \pm\bar{c}$, ψ, v, and u may all be discontinuous, and furthermore

$$\frac{\mathrm{d}v}{\mathrm{d}\psi} = -(u \pm \lambda\bar{c}) \tag{18}$$

$$\frac{\mathrm{d}u}{\mathrm{d}\psi} = v. \tag{19}$$

A second differentiation of equation (19) and substitution of equation (18) in the result leads to the requirements

$$u = P\sin\psi + Q\cos\psi \mp \lambda\bar{c} \tag{20}$$

$$v = P\cos\psi - Q\sin\psi \tag{21}$$

where P and Q are arbitrary constants.

In other words, longitudinal waves progress along lines $X = \pm ct$ and they are characterized by equation (17) and the conditions

$$\psi = \text{const.} \tag{22a}$$

$$v = \text{const.} \tag{22b}$$

Transverse waves progress along lines $X = \pm\bar{c}t$, and are characterized by equations (20) and (21), and the condition

$$\lambda = \text{const.} \tag{23}$$

3. EFFECTS OF A TRAVELING FORCE

Consider the force per unit undeformed cross-sectional area of the wire σ_T, traveling at speed V along the string. Let subscripts B and A refer to conditions before and after the force passes along a segment of the string and denote the undeformed cross-sectional area by a. Then, in time Δt, material in amount $\rho a V \Delta t$, under the influence of stresses σ_A, σ_B, and σ_T changes its velocity components from u_B, v_B to u_A, v_A; its orientation by angle ψ_{AB}, and its stretch ratio from λ_B to λ_A, as shown in Fig. 1.

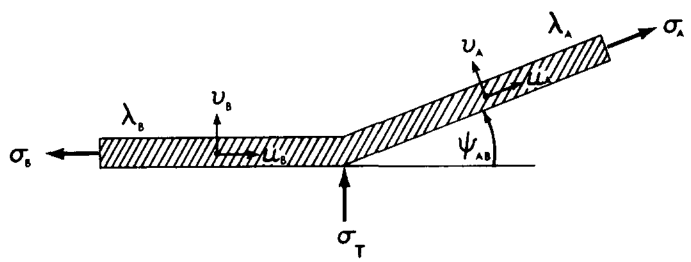

FIG. 1

Consideration of particle and wavefront displacements occurring in Δt leads to

$$u_B = (u_A + \lambda_A V)\cos\psi_{AB} - v_A \sin\psi_{AB} - \lambda_B V \tag{24}$$

$$v_B = (u_A + \lambda_A V)\sin\psi_{AB} + v_A \cos\psi_{AB}. \tag{25}$$

These equations could have been obtained by application of equations (20) and (21) first to one and then to the other side of the wavefront, since they are purely kinematic relations, unaffected by the presence of the transverse force.

Linear impulse–momentum considerations lead to

$$(\sigma_A - \sigma_B \cos\psi_{AB}) + \sigma_T \sin\psi_{AB} = \rho V(u_A - u_B \cos\psi_{AB} - v_B \sin\psi_{AB}) \tag{26}$$

$$\sigma_B \sin\psi_{AB} + \sigma_T \cos\psi_{AB} = \rho V(v_A - v_B \cos\psi_{AB} + u_B \sin\psi_{AB}) \tag{27}$$

Although these two equations are written specifically for the case that the traveling force is transverse to string segments immediately in front of it, they can be easily modified to account for a traveling force oriented arbitrarily in the plane of motion of the string.

4. CONSTRUCTION OF SOLUTIONS

The solution to the overall problem will depend on the relative magnitudes of the propagation speeds of the various waves which occur in the string. The propagation speed of the transverse wave accompanying each traveling force is prescribed by the burning rate of the explosive. It will be seen that a second transverse wave will also propagate in each direction from the point of ignition under the influence of no external forces. Its propagation speed is determined by the level of strain in the vicinity of the wavefront, as given by the second of equations (14). It may propagate either ahead of or behind the traveling force. One or more longitudinal shock waves may propagate, or a whole series of small longitudinal (plastic) wavelets may propagate, or a combination of these two. Longitudinal wave speeds are determined by the constitutive equation for the string material, as given by the first of equations (14).

In some cases the longitudinal wave is a wave of unloading, and this must be considered in the description of the stress–strain relation. For the more common engineering materials an increase in stress is propagated as a series of wavelets, and a decrease propagates in a shock wave. Longitudinal waves may propagate ahead of, behind, or mixed in with any of the transverse waves.

No matter what the ordering of the wave speeds, the initial conditions will be taken to be

$$u = v = \psi = 0 \quad \lambda = \lambda_0 \quad \text{for } \eta = \infty (t = 0)$$

and symmetry requirements dictate the boundary conditions

$$u = \psi = 0 \quad \text{for } \eta = 0\,(X = 0).$$

The point $X = 0$ is chosen to be the ignition point, and because of symmetry, it suffices to consider only waves progressing in the positive X direction.

Two examples of how the solution is constructed for a given ordering of wave speeds will be given below, and from these it can be seen how the solution would be constructed for any other ordering of wave speeds. In every case, conditions on the two sides of the transverse wave front accompanying the traveling wave must satisfy equations (20), (21), (26) and (27). The second transverse wave front is subject to equations (20), (21) and (23), and any longitudinal wavefront is subject to equations (17) and (22).

Commercially pure, annealed aluminum possesses an elastic longitudinal wave speed of approximately 2×10^5 in/sec. At a strain of 0·01, the plastic longitudinal and the transverse wave speeds are approximately 3×10^4 in/sec and 5×10^3 in/sec respectively. Kanninen and Florence [3] report detonation speeds in the range $(1{\cdot}2–2{\cdot}8) \times 10^5$ in/sec. The two examples have been chosen accordingly.

Example I: $V >$ *all* $c > \bar{c}$

The characteristics diagram appropriate to this situation is shown in Fig. 2(a) and the corresponding string configuration in Fig. 2(b). The characteristics are the lines η = constant, and only those corresponding to non-constant states are indicated. In constructing Fig. 2(a), the constancy of v and ψ across the longitudinal wavelets, the constancy of λ across the second transverse wave, and the initial and boundary conditions have been taken into account. The subscripts refer to the various constant states denoted by circled numbers. The problem reduces to finding $u_1, v_1, \psi_1, \lambda_1, u_2, \lambda_2$ and v_3 for given λ_0, σ_T, V and $\sigma(\lambda)$. c and $\bar{c}$ depend only on strain level for a given $\sigma(\lambda)$ through equations (14).

Evaluating P and Q in equations (20) and (21) from state 0, and using them for state 1 leads to

$$u_1 = \lambda_0 V \cos \psi_1 - \lambda_1 V \tag{28}$$

$$v_1 = -\lambda_0 V \sin \psi_1 . \tag{29}$$

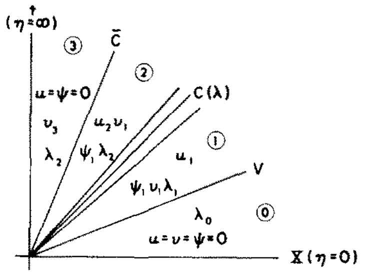

FIG. 2(a)

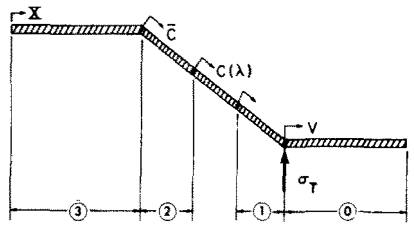

FIG. 2(b)

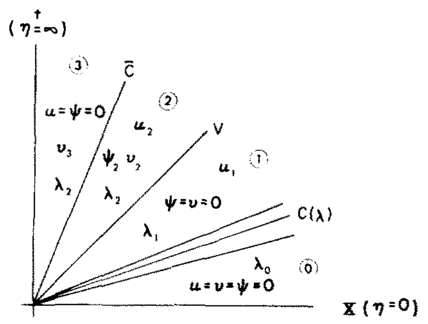

FIG. 3(a)

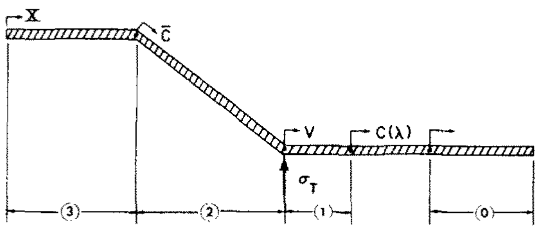

FIG. 3(b)

Evaluating equations (26) and (27) for states 0 and 1 leads to

$$(\sigma_1 - \sigma_0 \cos \psi_1) + \sigma_T \sin \psi_1 = \rho V u_1 \tag{30}$$

$$\sigma_0 \sin \psi_1 + \sigma_T \cos \psi_1 = \rho V v_1 . \tag{31}$$

Equation (17) relates states 1 and 2 by

$$u_2 = u_1 - \int_{\lambda_1}^{\lambda_2} c \, d\lambda \tag{32}$$

and equations (20) and (21) evaluated for states 2 and 3 yield

$$u_2 = v_3 \sin \psi_1 + \lambda_2 \bar{c}_2 (\cos \psi_1 - 1) \tag{33}$$

$$v_1 = v_3 \cos \psi_1 - \lambda_2 \bar{c}_2 \sin \psi_1 . \tag{34}$$

The above seven equations provide sufficient information for the solution to the problem. The required information may be put into more explicit form. Substituting equations (28) and (29) into (30) and (31) and solving for the unknown terms $\sigma(\lambda_1)$, λ_1, and ψ_1, obtain

$$\tan \psi_1 = -\frac{\sigma_T}{s_0} \tag{35}$$

$$s_1^2 = s_0^2 + \sigma_T^2 \tag{36}$$

where the notation has been used

$$s_i = \sigma_i + \rho \lambda_i V^2 \qquad i = 0, 1.$$

With these results, equations (28) and (20) become

$$s_1 u_1 = V(s_0\lambda_0 - s_1\lambda_1) \tag{37}$$

$$s_1 v_1 = \lambda_0 \sigma_T V \tag{38}$$

permitting equations (33) and (34) to be rearranged to

$$(s_1 - s_0)\lambda_2 \bar{c}_2 - s_0 u_{12} = V(s_1\lambda_0 - s_0\lambda_1) \tag{39}$$

$$s_0 v_3 = \sigma_T(\lambda_0 V - \lambda_2 \bar{c}_2) \tag{40}$$

where

$$u_{12} = -\int_{\lambda_1}^{\lambda_2} c \,\mathrm{d}\lambda$$

λ_2 is the only intrinsic unknown in the left hand side of equation (39).

As Kanninen and Florence explain, if every element of the string were to receive the velocity impulse simultaneously, neither longitudinal nor transverse waves would propagate, the tension would remain at its initial value, and the impulsive velocity achieved would be

$$\frac{\sigma_T}{\rho V}$$

Let R be the ratio of the final velocity achieved, v_3, to the above ideal value. Then, with the use of equation (40),

$$R = \frac{1 - \lambda_2 \bar{c}_2/\lambda_0 V}{1 + (\bar{c}_0/V)^2}.$$

If σ_T is small enough so that λ_2 is not very different from λ_0, this reduces to the value Kanninen and Florence found, provided that

$$R \doteq 1 - \frac{\bar{c}_0}{V}.$$

Illustrative sets of results are presented in Figs. 4 and 5 for this case. In Fig. 4, the stress–strain relation is taken to be an ideal bilinear hysteretic one with initial slope of 10^7 psi, secondary slope of $2{\cdot}5 \times 10^4$ psi, and a yield stress of 5×10^4 psi. Other relevant data are

$$\rho = 2{\cdot}5 \times 10^{-4}\ \text{lb-sec}^2/\text{in}^4$$

$$\varepsilon_0 = (\lambda_0 - 1) = 5 \times 10^{-3}$$

$$V = 2{\cdot}25 \text{ or } 2{\cdot}50 \times 10^5\ \text{in/sec.}$$

These results show that the longitudinal wave is a wave of unloading (that is, $\lambda_2 < \lambda_1$).

In Fig. 5, the stress–strain relation is taken to be linear elastic, with

$$E = 10^7\ \text{psi}$$

$$\rho = 2{\cdot}5 \times 10^{-4}\ \text{lb-sec}^2/\text{in}^4$$

$$\varepsilon_0 = 0$$

$$V = 2{\cdot}25 \text{ or } 2{\cdot}50 \times 10^5\ \text{in/sec.}$$

Here, the longitudinal wave is a loading wave.

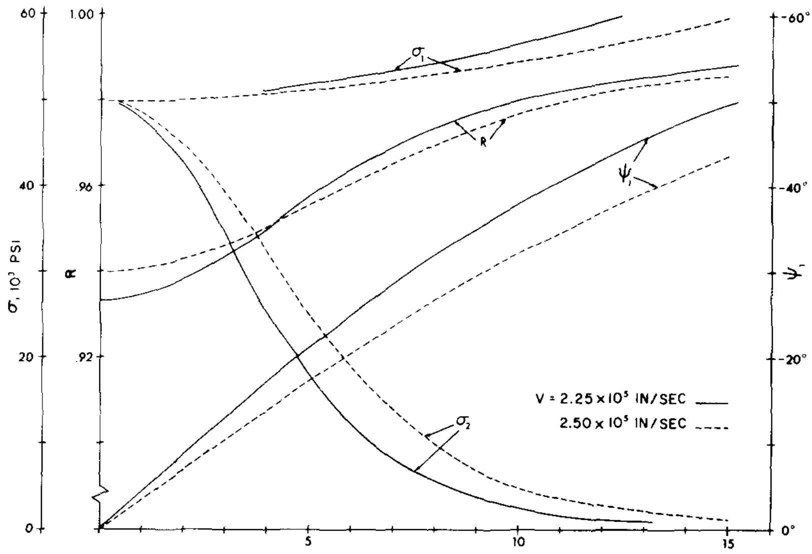

FIG. 4

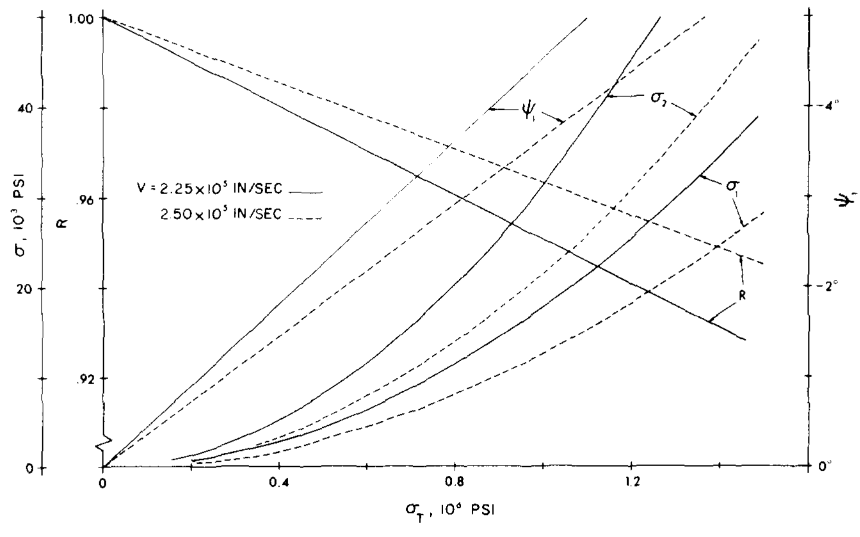

FIG. 5

In both of the above illustrations, the longitudinal wave is a shock wave because the stress–strain relation is a straight line between λ_1 and λ_2.

Example II: *all* $c > V > \bar{c}$

The characteristics diagram and string configuration are shown in Figs. 3(a) and 3(b), where again the initial and boundary conditions and the more simple requirements on changes in variables across wavefronts have already been accounted for. The problem then is to find $u_1, \lambda_1, u_2, \psi_2, v_2, \lambda_2$, and v_3 for given λ_0, σ_T, V and $\sigma(\lambda)$.

The seven equations needed for the solution are as follows. Equation (17) relates states 0 and 1 by

$$u_1 = -\int_{\lambda_0}^{\lambda_1} c\,\mathrm{d}\lambda. \tag{41}$$

States 1 and 2 are related by equations (20), (21), (26) and (27), which yield

$$u_2 = (u_1+\lambda_1 V)\cos\psi_2-\lambda_2 V \tag{42}$$

$$v_2 = -(u_1+\lambda_1 V)\sin\psi_2 \tag{43}$$

$$\sigma_2-\sigma_1\cos\psi_2+\sigma_T\sin\psi_2 = \rho V(u_2-u_1\cos\psi_2) \tag{44}$$

$$\sigma_1\sin\psi_2+\sigma_T\cos\psi_2 = \rho V(v_2+u_1\sin\psi_2) \tag{45}$$

Finally, states 2 and 3 are related by equations (20) and (21), which yield

$$u_2 = v_3\sin\psi_2+\lambda_2\bar{c}_2(\cos\psi_2-1) \tag{46}$$

$$v_2 = v_3\cos\psi_2-\lambda_2\bar{c}_2\sin\psi_2 \tag{47}$$

This solution may be expressed more conveniently as follows. With the notation

$$s_i = \sigma_i+\rho\lambda_i V^2 \qquad i = 1,2$$

equations (42)–(45) may be rearranged to

$$\sigma_T^2+s_1^2 = s_2^2 \tag{48}$$

$$\tan\psi_2 = -\sigma_T/s_1 \tag{49}$$

$$s_2(u_2+\lambda_2 V) = s_1(u_1+\lambda_1 V) \tag{50}$$

$$s_2 v_2 = (u_1+\lambda_1 V)\sigma_T. \tag{51}$$

Using these results, equations (46) and (47) may be combined to obtain

$$s_1 v_3 = \sigma_T(u_1+\lambda_1 V-\lambda_2\bar{c}_2). \tag{52}$$

Equations (48), (51) and (52) may be combined to

$$(u_1+\lambda_1 V)s_2 = \lambda_2[s_1 V+\bar{c}_2(s_2-s_1)] \tag{53}$$

with the result that equations (41), (48) and (53) must be simultaneously solved for the three implicit unknowns; λ_1, u_1 and λ_2. Then ψ_2, u_2, v_2 and v_3 may be found from equations (49), (50), (51) and (52).

In this case, the ratio of the final velocity achieved to its ideal value, which may be obtained by combining equations (52) and (53), is

$$R = \frac{1-\bar{c}_2/V}{1+(\bar{c}_2/V)^2}$$

Figure 6 presents illustrative results for the case $V < c$. The data for Fig. 5 are used here also, except

$$V = 1{\cdot}0 \text{ or } 1{\cdot}5\times 10^5 \text{ in/sec.}$$

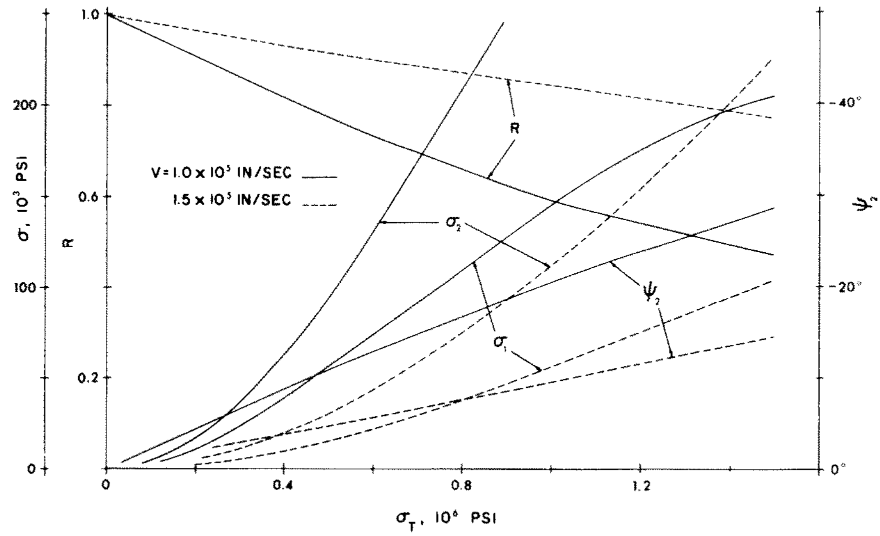

FIG. 6

Again, the second longitudinal wave is a loading wave, and because of the linear relationship between stress and strain, it is a shock wave.

5. DISCUSSION

Despite the fact that the solution to the title problem has been found in closed form, it would require extensive calculation to determine completely how changing the input parameters (σ_T, V, λ_0, $\sigma(\lambda)$) affects the response of the string. In particular, the constitutive equation determines the various wave speeds, and the ordering of these determines which set of equations apply.

The responses illustrated in Figs. 4, 5 and 6 were selected to give at least some insight into the effects of string extensibility. In the case of Fig. 4, extensibility leads to velocity simulation closer to the ideal ($R = 1$) than would occur in an inextensible string. However, the final velocity may be achieved much more slowly than that analysis indicates, since $\bar{c}_2$ becomes considerably smaller than $\bar{c}_0$ for large σ_T. In addition, the inextensible model does not predict the large strains (as much as 0·40 for these input parameter ranges) which occur.

The contrast between the responses illustrated in Figs. 5 and 6 is marked. In both these cases, $\bar{c}_0 = 0$, and the inextensible model would indicate V is highly supersonic and therefore, simulation excellent. Relative to the longitudinal wave speed, V is supersonic in Fig. 5 and subsonic in Fig. 6. In the former case simulation is very good, but the latter case shows simulation to be poor, with the string stressed to levels that would cause most materials to fail. Therefore, there are at least some circumstances in which the use of small-deformation theory is inadequate to describe the response.

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Абстракт—Исследуется бесконечно длинный, идеально гибкий стержень, подверженный нагрузке двумя сосредоточенными силами, которые передвигаются с постоянной скоростью вдоль стержня, в противоположенных направлениях, исходящих из одной точки. Описываются создавшиеся движения и деформации стержня. Расчет иринимает во внимание большие деформации, изменения в растяжении стержня, а также распределение поперечных длинных волн.